Additive Hermitian idempotent preservers*

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Notation, Definition

Notation: M_n: set of $m \times n$ real or complex matrices
$H_n = \{A \in M_n : A = A^*\}$: set of Hermitian matrices
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Definition:

1. $A, B \in M_n$ are orthogonal, denoted by $A \perp B$, if $A^*B = AB^* = 0$.
   Hence A and B have disjoint ranges and kernels.
2. $A \in M_n$ is Hermitian if $A^* = A$.
2. $A \in M_n$ is idempotent if $A^2 = A$.
3. $A \in M_n$ is projection if $A^2 = A = A^*$.

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Notation. $M_n(\mathbb{F})$: set of $n \times n$ matrices over field $\mathbb{F}$ (character. not 2).

Theorem (Frobenius, 1897)

Let $L : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map satisfying $\det(L(A)) = \det A$
$\Leftrightarrow$ $\exists$ invertible matrices $M, N \in M_n(\mathbb{C})$ with $\det(MN) = 1$ s.t.
$L(A) = MAN \quad \forall A \in M_n(\mathbb{C})$ or $L(A) = MA^tN \quad \forall A \in M_n(\mathbb{C})$.

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Definition: Let $L : M_n \to M_m$ be an additive map.

1. L is a Jordan homomorphism if $L(A^2) = L(A)^2$, or equivalently,
$$L(AB + BA) = L(A)L(B) + L(B)L(A) \qquad \forall\, A, B \in M_n$$
2. L is an algebra or a ring or a Jordan *-homomorphism if L is an algebra, a ring or a homomorphism with $L(A^*) = L(A)^*$

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Suppose $L : M_n \to M_m$ is a linear Jordan homomorphism. Then $\exists\, p, q \in \mathbb{Z}^+$ with $n(p+q) = r \le m$, s.t. L has the form

$$A \mapsto S \begin{pmatrix} I_p \otimes A & & \\ & I_q \otimes A^t & \\ & & 0_{m-r} \end{pmatrix} S^{-1} \qquad \text{for some invertible } S \in M_m.$$

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Theorem (Cho, Zhang, 1996)

$\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. An additive map $L : M_n \mapsto M_n$ sending idem. to idem. Then
(1) $L(A) = \sigma(trA)$, for an additive $\sigma : \mathbb{F} \to M_m$ with $\sigma(1) = 0$.
(2) $L(A) = S(A_\tau + \sigma(trA))S^{-1}$, for injective endomorphism τ on $\mathbb{F}$ with $\tau(1) = 1$, invertible $S \in M_n$, σ in (1). Here $A_\tau = (\tau(a_{ij}))$ if $A = (a_{ij})$.
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Note. L is linear $\Rightarrow$ $A - \lambda B$: idem. $\Leftrightarrow L(A) - \lambda L(B)$: idem.

Theorem (Semrl, 2003)

A bijective continuous map $L : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying

$$A - \lambda B: \text{idem.} \Leftrightarrow L(A) - \lambda L(B): \text{idem.} \qquad \forall\, A, B \in M_n(\mathbb{C}),\ \lambda \in \mathbb{C}$$

$\Leftrightarrow \exists$ invertible matrix $S \in M_n$ s.t.

$$L(A) = SAS^{-1} \quad \forall\, A \in M_n(\mathbb{C}) \quad \text{or} \quad L(A) = SA^tS^{-1} \qquad \forall\, A \in M_n(\mathbb{C}).$$

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An additive map $L : H_n \to M_m$, sending (rank≤ 2) Herm. idem. to Herm. idem. $\Leftrightarrow \exists\, p, q \ge 0$, $n(p+q) = r \le m$, unitary $U \in M_m$ s.t.

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Example

Note: 1. An additive map $L : H_n \to M_m$ is exactly rational linear, but may not be real linear.

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$\Leftrightarrow$ (1) $L(\alpha I_n) = U(\alpha I_{p+q} \oplus 0)U^* \quad \forall$ rational number α, and

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Ex: Define an additive map $T : H_n \to M_m$ by $T(A) := L(A) + L_0(A)$, where L_0 is any additive map, s.t. $L_0(A) = 0$ when $trA = 0$. For example, $L_0(A) = \sigma(trA)$, where $\sigma : \mathbb{R} \to M_m$ is any rational linear map, s.t. $\sigma(1) = 0$. Then

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Lemma 1, Lemma 2

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If $L : H_n \to M_m$: additive, preserves Herm. idem., then

$$L(P) \leq L(Q) \quad \forall \text{ Herm. idem. } P, Q \in H_n, P \leq Q.$$

Pf. $\exists\, \hat{P}$, s.t. $Q = P + \hat{P}$, $\hat{P} \perp P$ $\therefore L(Q) = L(P) + L(\hat{P}) \geq L(P)$.

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If $L : H_n \to M_m$ is additive, sending (rank≤ 2) Herm. idem. to Herm. idem., then L is real linear on the set of trace zero Hermitian matrices.

Pf: 1. Let $\{E_{ij}\}$ be standard basis of M_n. For $i \neq j$,
$\because L(E_{ii} + E_{jj}) = L(E_{ii}) + L(E_{jj})$ is Herm. idem., $\therefore L(E_{ii}) \perp L(E_{jj})$.
We may assume that $\exists\, U \in U_m$, s.t.
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May assume that $U = I_m$, $i = 1, j = 2$.

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Proof of Lemma 2

2. Let $\alpha_n > 0$, $\{\alpha_n\} \to 0$, $Q_n = \alpha_n(E_{11} - E_{22})$,
   $\forall\, n$, let r_n with $0 < \alpha_n < r_n \leq 2\alpha_n$,
   $R_n = \begin{pmatrix} r_n & 0 \\ 0 & r_n \end{pmatrix} \oplus 0$, $A_n^{\pm} = \begin{pmatrix} \alpha_n & \pm\sqrt{r_n^2 - \alpha_n^2} \\ \pm\sqrt{r_n^2 - \alpha_n^2} & -\alpha_n \end{pmatrix} \oplus 0$.
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   Hence $L(\alpha_n(E_{ii} - E_{jj})) \to 0 \;\; \forall\, i \neq j$.

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3. $\forall\, A = E_{ii} - E_{jj} \in H_n$, $i \neq j$, $\because \alpha > 0$, $\exists\, q_n > 0$, $q_n \to \alpha$.
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Sketch proof

Theorem (LTWW)

An additive map $L : H_n \to M_m$, sending (rank$\leq$ 2) Herm. idem. to Herm. idem. $\Leftrightarrow \exists\, p, q \geq 0$, $n(p+q) = r \leq m$, unitary $U \in M_m$ s.t.

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Main idea of $\Rightarrow$: Note that $\forall\, A \in H_n$, $A = [A - tr(A)I_n] + tr(A)I_n$.

Define a real linear map $\tilde{L} : H_n \to M_m$ by

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1. Assume that $\mathbb{F} = \mathbb{C}$. Define a complex linear map $\tilde{L} : M_n \to M_m$ by $\tilde{L}(H + iG) = \tilde{L}(H) + i\tilde{L}(G)$ for $H, G \in H_n$.

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$\tilde{L} : A \mapsto U[(I_p \otimes A) \oplus (I_q \otimes A^t) \oplus 0]U^* \quad \forall\, A \in M_n$.
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4. Assume that $\mathbb{F} = \mathbb{R}$. We use other approach.
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   $\therefore \tilde{L}$ preserves zero products ($AB = 0 \Rightarrow \tilde{L}(A)\tilde{L}(B) = 0$)
   May assume $\tilde{L}(I_n) = I_s \oplus 0_{m-s}$. Moreover, assume $m = s$, $\tilde{L}(I_n) = I_s$
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6. Let $B = \tilde{L}(E_{12} + E_{21}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0$, $B_1 \in M_{k_1,k_1}$, $B_2 \in M_{k_2,k_2}$.
   Choose $X_1 = \begin{pmatrix} \gamma & 1 \\ 1 & 1/\gamma \end{pmatrix} \oplus 0$, $X_2 = \begin{pmatrix} 1/\gamma & -1 \\ -1 & \gamma \end{pmatrix} \oplus 0$, $\gamma \in \mathbb{R}$,
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Corollary

Corollary (LTWW)

An additive map $L : M_n \to M_m$, sending (rank≤ 2) Herm. idem. to Herm. idem. $\Leftrightarrow \exists\, p, q \geq 0$, $n(p+q) = r \leq m$, unitary $U \in M_m$ s.t.

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Definition 1. $P(M_n)$: the lattice of $n \times n$ Herm. idem..

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Note. 1. $span_Q P(M_n)$ consists of $n \times n$ Hermitian matrices with rational trace. (Fillmore, 1969)

($A \in M_n(\mathbb{C})$ is sum of proj. $\Leftrightarrow A \geq 0$, $tr(A)$ is integer, $tr(A) \geq$ rank$A$)

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Let $n > 1$, K: Hilbert space. $L : H_n \to B(K)$: a nonzero additive Herm. idem. preserver. Then $\dim K \geq n$, & $\exists$ unitary U, orthogonal Herm. idem. $I_r, I_s \in B(K)$ s.t.

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H: separable complex Hilbert space. $L : B(H)_{\mathrm{sa}} \to B(H)$: a nonzero additive map preserving Herm. idem.. Then $\exists$ unitary U, orthogonal projections $I_r, I_s \in B(K)$ s.t.

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Definition: 1. $B(H)$: bounded linear operators on Hilbert space H.
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Theorem (Dye, 1955)

M, N: von Neumann algebras s.t. M does not contain a direct type I_2 summand, then any bijective map $L : P(M) \to P(N)$ sending orthogonal Herm. idem. to orthogonal Herm. idem. extends uniquely to a Jordan $*$-isomorphism between the whole algebras.

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Theorem

Let M be a von Neumann algebra and B be a C^*-algebra. Let $L : span_Q P(M) \to span_Q P(B)$ be an additive map sending Herm. idem. to Herm. idem.. Then L extends to a Jordan $*$-homomorphism $J : M \to B$.

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Thanks for your attention !

$M \cong \bigoplus_j M_n(L^\infty(\Omega_j, \mu_j))$ is a finite type I_n factor

Theorem

Let B be a C^*-algebra, let (Ω, μ) be a measure space and let n be an integer $n \geq 2$. Let $L : M_n(L^\infty(\Omega, \mu))_{\mathrm{sa}} \to B$ be an additive map sending Herm. idem. to Herm. idem.. Then there is a Jordan $*$-homomorphism $J : M_n(L^\infty(\Omega, \mu)) \to B$ such that $J(A) = L(A)$ whenever $A \in span_Q P(M_n(L^\infty(\Omega, \mu)))$.

Proposition

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Let $\theta : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a complex linear map. Then
1. θ is Jordan homomorphism $\Leftrightarrow$ θ sends idempotents to idempotents.
2. θ is Jordan *-homomorphism $\Leftrightarrow$ θ sends projections to projections.

Main idea of 1($\Leftarrow$) : Assume θ sends idempotents to idempotents.

(1) Show θ sends disjoint idempotents to disjoint idempotents.

(2) Show $\theta(A^2) = (\theta(A))^2$ if $A = A^* \in M_n$

(3) Show $\theta(BC + CB) = \theta(B)\theta(C) + \theta(C)\theta(B)$ if $B = B^*, C = C^* \in M_n$

(4) Show $\theta(A^2) = (\theta(A))^2$ for any $A \in M_n$

Therefore θ is a Jordan homomorphism.

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Pf of 1($\Leftarrow$) : Assume θ sends idempotents to idempotents.
(1) Check θ sends disjoint idempotents to disjoint idempotents.
Suppose that P, Q are two idempotents s.t. $PQ = QP = 0$. Then $P + Q$ is also an idempotent, and thus
$\theta(P+Q) = (\theta(P)+\theta(Q))^2 = \theta(P) + \theta(P)\theta(Q) + \theta(Q)\theta(P) + \theta(Q)$.
Hence $\theta(P)\theta(Q) = -\theta(Q)\theta(P)$
$\Rightarrow \theta(P)\theta(Q) = \theta(P)^2\theta(Q) = -\theta(P)\theta(Q)\theta(P) = \theta(Q)\theta(P)^2 = \theta(Q)\theta(P) = 0$.

Pf of 1($\Leftarrow$) : Show θ is a Jordan homomorphism.
(1) θ sends disjoint idempotents to disjoint idempotents.
(2) Check $\theta(A^2) = (\theta(A))^2$ if $A = A^*$
Let $A = A^* \in M_n$. Then $A = \sum_k \lambda_k P_k$ for some orthogonal projections P_k. Hence $A^2 = \sum_k \lambda_k^2 P_k$, and thus

$$\theta(A^2) = \sum_k \lambda_k^2 \theta(P_k) = (\theta(A))^2.$$

(3) Check $\theta(A^2) = (\theta(A))^2$ for any $A \in M_n$
Let $A = B + \mathrm{i}C$ with $B = B^*, C = C^* \in M_n$. Since $(B + C) = (B + C)^*$, we have $\theta((B + C)^2) = (\theta(B) + \theta(C))^2$. Hence
$\theta(BC + CB) = \theta(B)\theta(C) + \theta(C)\theta(B)$. Thus $A^2 = B^2 + i(BC + CB) - C^2$.
Therefore

$$\theta(A^2) = (\theta(B))^2 + \mathrm{i}(\theta(B)\theta(C) + \theta(C)\theta(B)) - (\theta(C))^2 = (\theta(A))^2.$$

Zero product preserver from S_n to M_r

Notation. $H_n(\mathbb{C})$ the real linear space of self-adjoint matrices in $M_n(\mathbb{C})$

Theorem

Let $\Phi : S_n \mapsto M_r$ be a linear map. Then T.F.A.E.

(1) $\Phi(A)$ is idempotent whenever A is idempotent with rank$A = 1 \& \Phi$ preserves zero products.

(2) $\exists\ k \in \mathbb{N}$, invertible $S \in M_r$ s.t. Φ has the form $A \mapsto S^{-1} \begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_{r-kn} \end{pmatrix} S$

Furthermore, if $\Phi(A) = \phi(A)^{\mathrm{t}}$ (for all idempotent) matrices $A \in M_n$, then S can be chosen to be complex orthogonal.